Symmetric Powers

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In this seminar talk we introduce the d-th symmetric power in the category of schemes and construct an isomorphism between the hilbert scheme $Hilb^d_{X/S}$ of X/S and the d-th symmetric power $(X/S)^{(d)}$ of X/S under suitable assumption on the scheme X/S. The talk presents [GW10, Proposition 12.27][1], [Gro71, Exposé V.1, Proposition 1.1,1.8][2] and [BLR90, Chapter 9.3, Proposition3] [3].

1 The Quotient of a Scheme by a Group

Definition 1.1. *Let X be a scheme, G be a group of automorphisms of X and $p : X \to Y$ a morphism of schemes such that $p = p \circ g$ for all $g \in G$. Then (Y, p) is called the* ***quotient of*** X ***by*** G *if for all morphisms of schemes $f : X \to Z$ such that $f = f \circ g$ for all $g \in G$ there exist a unique morphism $\bar{f} : Y \to Z$ with $\bar{f} \circ p = f$. The quotient is also denoted by X/G.*

Remark 1.2. *By definition, if $X \to S$ is a morphism of schemes and $G \subseteq Aut_{(Sch/S)}(X/S)$ then Y is a scheme over S st. $p : X \to Y$ is a morphism in (Sch/S).*

Example 1.3. *If $X/S \in (Sch/S)$ then an element σ in the symmetric group S_d yields an automorphism of the d-th fibre product $(X/S)^d$ of X over S given by*

$$(X/S)^d(T) \to (X/S)^d(T) \ ; \ (t_1, \ldots, t_d) \mapsto (t_{\sigma(1)}, \ldots, t_{\sigma(d)})$$

If the quotient of $(X/S)^d$ by S_d exists then we call it the ***d-th symmetric power*** *of X/S and denote it as $(X/S)^{(d)}$.*

Theorem 1.4. *Let $X = SpecA$ and G be a finite group of automorphisms of X and denote the corresponding morphism of rings for $g \in G$ as $g^{-1} : A \to A$. Furthermore, set $A^G = \{a \in A | a = g(a) \text{ for all } g \in G\}$ and $Y = SpecA^G$. Then (Y, p) is the quotient of X by G, where $p : SpecA \to SpecA^G$ corresponds to the inclusion map on rings.*

Example 1.5. $(\mathbb{A}^1_{\mathbb{Z}}/Spec\mathbb{Z})^{(d)} = Spec(\mathbb{Z}[X_1, \ldots, X_d]^{S_d})$ *is isomorphic to $Spec(\mathbb{Z}[e_1, \ldots, e_d])$ with e_k mapping to elementary symmetric polynomials $\sum\limits_{1 \le i_1 < \cdots < i_k \le d} x_{i_1} \ldots x_{i_k}$.*

Proposition 1.6. *Let X be a scheme, G be a finite group of automorphisms of X and $p : X \to Y$ be an affine morphism st. $p(g(x)) = p(x)$, for $g \in G$ and $x \in X$, and p^b yields an isomorphism $\mathcal{O}_Y \to p_*\mathcal{O}_X^G$ with $p_*\mathcal{O}_X^G = ker(p_*\mathcal{O}_X \to \prod_{g \in G} p_*\mathcal{O}_X \ ; \ a \mapsto \prod_{g \in G}(a - g(a)))$.*

Theorem 1.7. *Let X be a scheme and G be a finite group of automorphisms of X. Then the following are equivalent:*

(i) There exists a morphism of schemes $p : X \to Y$ as in Proposition 1.6.

(ii) For $x \in X$ there exists an open affine subset $U \subseteq X$ containing the orbit $xG = \{g(x) | g \in G\}$ of x.

(iii) It exists an open affine cover of $X = \bigcup_{i \in I} U_i$ st. $g(U_i) \subseteq U_i$, for $g \in G$.

Corollary 1.8. *If X/S is quasi-projective then the d-th symmetric power $(X/S)^{(d)}$ of X/S exists.*

[1] [GW10] U.Görtz and T. Wedhorn, Algebraic Geometry I: Schemes, Springer Spektrum, Wiesbaden, 2010

[2] [Gro71] A. Grothendieck, Revêtment étales et groupe fondamental (SGA1), Lecture Notes in Mathematics vol. 224, Springer, 1971

[3] [BLR90] S. Bosch, W. Lütkebohmert, and M. Raynaud, Néron models, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, vol. 21, Springer Berlin & Heidelberg, 1990

Lemma 1.9. *If X/S is quasi-projective and flat then*

(i) the d-th symmetric power $(X/S)^{(d)}$ is flat over S and

(ii) for $T \in (Sch/S)$, we have $(X/S)^{(d)} \times_S T \cong (X_T/T)^{(d)}$.

2 The Hilbert scheme and the Symmetric Power

In the previous talk we have constructed a map

$$(X/S)^d(T) \to (X_T/T)^d(T) \to Hilb^d_{X/S}(T)$$

$$(t_1, \dots, t_d) \mapsto (t_{1_T}, \dots, t_{d_T}) \mapsto t_{1_T}(T) + \dots + t_{d_T}(T)$$

which is surjective and finite locally free of rank $d!$. Furthermore, it is symmetric and thus if X/S is also quasi-projective then the above map factors into

$$(X/S)^d \overset{\pi}{\to} (X/S)^{(d)} \overset{\alpha}{\to} Hilb^d_{X/S}$$

In the following we will construct an inverse of α.

Definition 2.1. *Let A be a ring and M, N be A-modules. A* ***polynomial law from*** *M* ***to*** *N is the collection of morphisms of sets $\{f_{A'} : M \otimes_A A' \to N \otimes_A A' | A' \text{ A-algebra}\}$ such that for all $\phi : A' \to A''$ morphisms of A-algebras we have $(id_N \otimes \phi) \circ f_{A'} = f_{A''} \circ (id_M \otimes \phi)$. If it exists $d \geq 0$ such that for any A' A-algebra, $a \in A'$ and $x \in M \otimes_A A'$ it holds that $f_{A'}(ax) = a^d f_{A'}(x)$ then $\{f_{A'} : M \otimes_A A' \to N \otimes_A A' | A' \text{ A-algebra}\}$ is called* ***homogeneous polynomial law of deg.*** *d.*

Remark 2.2. *If we assume that M and N are finite free k-modules then there is the following bijection:*[4]

$$\{\text{polynomial laws from } M \text{ to } N\} \overset{1:1}{\leftrightarrow} \{\text{polynomial maps from } M \text{ to } N\}$$

Example 2.3. *Let A be a ring and B a flat A-algebra. Then the collection of maps $\gamma^d_{A'} : B \otimes_A A' \to (B \otimes_A \cdots \otimes_A B)^{S_d} \otimes_A A' \cong ((B \otimes_A A') \otimes_{A'} \cdots \otimes_{A'} (B \otimes_A A'))^{S_d}$ given by $x \mapsto x \otimes \cdots \otimes x$, for A' an A-algebra, yields a homogeneous polynomial law of degree d from B to the d-th symmetric tensor power of B over A.*

Remark 2.4. *Consider $A, B, \gamma^d_{A'}$ as in example 2.3. Then $\{\gamma^d_{A'} | A' \text{ A-algebra}\}$ is* ***universal****, i.e. for all homogeneous polynomial laws $\{f_{A'} | A' \text{ A-algebra}\}$ of degree d from B to A, it exists a unique A-linear map $\phi : (B \otimes_A \cdots \otimes_A B)^{S_d} \to A$ such that $f_{A'} = (\phi \otimes A') \circ \gamma^d_{A'}$ with A' an A-algebra.*

Example 2.5. *Let A be a ring, B be a flat A-module and $\mathcal{L}$ be a B-module that is free of rank d over A. Then we denote the determinant of the corresponding matrix with coefficients in A of $\mathcal{L} \overset{\cdot b}{\to} \mathcal{L}$ the multiplication with $b \in B$ as $det_{\mathcal{L}}(b)$. Now,*

$$\{det_{\mathcal{L},A'} : B \otimes_A A' \underset{x \mapsto det_{\mathcal{L} \otimes_A A'}(x)}{\to} A \otimes_A A' | A' \text{ A-algebra}\}$$

is a homogeneous polynomial law of degree d from B to A. As such and the fact that γ^d is universal we get a unique A-linear morphism $\det_{\mathcal{L}} : (B \otimes_A \cdots \otimes_A B)^{S_d} \to A$ such that $det_{\mathcal{L},A'} = (det_{\mathcal{L}} \otimes A') \circ \gamma^d_{A'}$ with A' an A-algebra. Moreover, $det_{\mathcal{L}}$ is a morphism of A-algebras.

Proposition 2.6. *Let X/S be quasi-projective and flat. Then there exists a canonical morphism of schemes $\sigma : Hilb^d_{X/S} \to (X/S)^{(d)}$.*

[4]see F. Vaccarino, Homogeneous multiplicative polynomial laws are determinants, Jornal of Pure and Applied Algebra, Volume 213, 2009

On T-valued points this morphism is given by

$$\begin{aligned} Hilb^d_{X/S}(T) &\to (X/S)^{(d)}(T) \\ D &\mapsto \sigma_T(D) \end{aligned}$$

with $\sigma_T(D) : T \to (D/T)^{(d)} \to (X_T/T)^{(d)} \to (X/S)^{(d)}$ and $T \to (D/T)^{(d)}$ is constructed as above (i.e. for $V \subseteq T$ open and affine st. $f_*\mathcal{O}_D(V)$ is free of rank d we consider $B = f_*\mathcal{O}_D(V), L = f_*\mathcal{O}_D(V)$ and $A = \mathcal{O}_T(V)$)

Theorem 2.7. *If X/S is smooth, proper, quasi-projective and of rel. dimension 1 then α and σ are inverse isomorphisms.*

This follows from $\sigma \circ \alpha = id$ which can be shown by induction on d and the following lemma.

Lemma 2.8. *Consider X/S as in theorem 2.7. Then*

(i) $(X/S)^{d_1} \times_S (X/S)^{d_2} \to (X/S)^{(d_1+d_2)}$ *implies a map* $d : (X/S)^{(d_1)} \times_S (X/S)^{(d_2)} \to (X/S)^{(d_1+d_2)}$

(ii) For $i = 1, 2$ and T-valued points $D_i \in Hilb^{d_i}_{X/S}(T)$ we have $\sigma_T(D_1 + D_2) = d(\sigma(D_1), \sigma(D_2))$.